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2005 J. Phys. A: Math. Gen. 38 5011

(http://iopscience.iop.org/0305-4470/38/22/022)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **38** (2005) 5011–5022 doi:10.1088/0305-4470/38/22/022

Wave equations with concentrated nonlinearities

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Received 3 November 2004, in final form 30 December 2004
Published 18 May 2005
Online at stacks.iop.org/JPhysA/38/5011

Abstract

In this paper, we address the problem of wave dynamics in the presence of concentrated nonlinearities. Given a vector field V on an open subset of $\mathbb{C}^n$ and a discrete set $Y \subset \mathbb{R}^3$ with n elements, we define a nonlinear operator $\Delta_{V,Y}$ on $L^2(\mathbb{R}^3)$ which coincides with the free Laplacian when restricted to regular functions vanishing at Y, and which reduces to the usual Laplacian with point interactions placed at Y when V is linear and represented by a Hermitian matrix. We then consider the nonlinear wave equation $\ddot{\phi} = \Delta_{V,Y}\phi$ and study the corresponding Cauchy problem, giving an existence and uniqueness result when V is Lipschitz. The solution of such a problem is explicitly expressed in terms of the solutions of two Cauchy problems: one relative to a free wave equation and the other relative to an inhomogeneous ordinary differential equation with delay and principal part $\dot{\zeta} + V(\zeta)$. The main properties of the solution are given and, when Y is a singleton, the mechanism and details of blow-up are studied.

PACS numbers: 03.65.Ge, 02.30.Tb

1. Introduction

In recent times a great effort has been devoted to the analysis of nonlinear wave equations. Among the more interesting themes, there are global existence, presence of blow-up solutions and characterization of their lifespan (see, e.g., [5, 6, 8, 9, 11, 13, 17] and references therein). These issues are usually quite difficult to analyse due to the scarcity of information about exact solutions of nonlinear wave equations. In this paper, we study a class of wave equations about which information on exact solutions is relatively easy to obtain. This class is characterized by a so-called concentrated nonlinearity, modelled as a nonlinear point interaction in some fixed finite set of points. To be more precise, we will study abstract wave equations of the form $\ddot{\phi} = \Delta_{V,Y}\phi$, where $\Delta_{V,Y}$ is a nonlinear operator on $L^2(\mathbb{R}^3)$ which coincides with the free Laplacian when restricted to regular functions vanishing at the point of $Y = \{y_1, \ldots, y_n\}$, a discrete and finite subset of $\mathbb{R}^3$. $V \equiv (V_1, \ldots, V_n)$ is a vector field on $\mathbb{C}^n$ which is related to the behaviour, at Y, of the functions ϕ belonging to the domain of $\Delta_{V,Y}$ by

$$\phi(x) = \frac{\zeta_j^\phi}{4\pi|x - y_j|} + V_j(\zeta^\phi) + O(|x - y_j|), \qquad 1 \leqslant j \leqslant n, \quad x \to y_j.$$

The action of $\Delta_{V,Y}$ can then be defined in a suggestive way by

$$\Delta_{V,Y}\phi := \Delta\phi + \sum_{1\leqslant j\leqslant n} \zeta_j^\phi \delta_{y_j},$$

where δ_y is the Dirac mass at y. We refer to section 2 for the precise definitions. Such a nonlinear operator reduces to the self-adjoint operator given by the Laplacian with n point interactions (see [3, 4]) when $V(\zeta) = \Theta\zeta$, Θ a Hermitian matrix.

In section 3 we then turn to the problem of existence and uniqueness of the Cauchy problem for the nonlinear wave equation $\ddot{\phi} = \Delta_{V,Y}\phi$. The analogous problem for the nonlinear Schrödinger equation $i\dot{\psi} = -\Delta_{V,Y}\psi$ was studied in [1, 2] in the case of particular nonlinearities of the kind $V_j(\zeta) = \gamma_j|\zeta_j|^{2\sigma_j}\zeta_j$, $\gamma_j \in \mathbb{R}$, $\sigma_j \geqslant 0$, whereas the wave equation case was studied, when V is linear, in [14–16, 7] when Y is a singleton and in [12] in the general case. The nonlinear wave equation case was instead totally unexplored. Thus in theorem 3.1 we provide an existence and uniqueness result when V is Lipschitz. The strategy of the proof follows the lines of the linear case with the complication due to the lack of a general existence theorem in this singular situation. Similar to the linear case, the main result is the relation between the equation $\ddot{\phi} = \Delta_{V,Y}\phi$ and a coupled system comprising an ordinary wave equation with delta-like sources and an inhomogeneous ordinary differential equation with delay driven by the vector field V. This delayed equation controls the dynamics of the coefficients ζ^ϕ and an almost complete decoupling is achieved, in that it is possible to get the solution of the problem when the retarded Cauchy problem for the ζ^ϕ (depending in a parametric way on the initial data of the field ϕ) is solved, except a term which is the free wave evolution of the initial data. A similar situation appears (for the particular nonlinearities indicated above) in the Schrödinger case (see [1]) where however, due to the infinite speed of propagation of the free Schrödinger equation, an integral Volterra-type equation replaces the ordinary differential equation.

When the vector field V is of gradient type, a conserved energy for the dynamics exists (see lemma 3.7) and this provides criteria for global existence (see theorem 3.8). When there is no global existence, the problem of the characterization of blow-up solutions and their blow-up rates arises. In the special case where the singularity is at only one point y, a detailed study is possible (see section 4). The key remark is that the inhomogeneous term in the equation for the ζ^ϕ is bounded and continuous and there exist simple autonomous first-order differential equations, the solution of which provides supersolutions and subsolutions by means of differential inequalities. This allows us to prove in many cases the existence at large or in contrast the blow-up of the solutions together with, in the latter case, an estimate of their lifespan. A typical example is a power law nonlinearity, where explicit calculations are given and in particular for the quadratic nonlinearity, which leads to an equation of Riccati type.

2. Nonlinear point interactions

Given the vector field $V : A_V \subseteq \mathbb{C}^n \to \mathbb{C}^n$, A_V open, and a discrete set $Y \subset \mathbb{R}^3$, $Y = \{y_1, \ldots, y_n\}$, we give the definition of a nonlinear operator $\Delta_{V,Y}$ on $L^2(\mathbb{R}^3)$ which reduces to the usual Laplacian with point interactions at Y when V is linear and is represented by a Hermitian matrix.

Definition 2.1. *We define the nonlinear subset $D_{V,Y}$ of $L^2(\mathbb{R}^3)$ by the set of $\phi \in L^2(\mathbb{R}^3)$ for which there exists an n-tuple of complex numbers $\zeta^\phi = \left(\zeta_1^\phi, \dots, \zeta_n^\phi\right) \in A_V$ such that*

$$\phi_{\mathrm{reg}} \in \bar{H}^2(\mathbb{R}^3) := \left\{ f \in L^2_{\mathrm{loc}}(\mathbb{R}^3) : \nabla f \in L^2(\mathbb{R}^3),\, \Delta f \in L^2(\mathbb{R}^3) \right\},$$

where

$$\phi_{\mathrm{reg}} := \phi - \sum_{1 \leqslant j \leqslant n} \zeta_j^\phi G_j, \qquad G_j(x) := \frac{1}{4\pi |x - y_j|},$$

and moreover the following nonlinear boundary conditions hold true at Y,

$$\lim_{x \to y_j} \left(\phi(x) - \zeta_j^\phi G_j(x) \right) = V_j(\zeta^\phi), \qquad 1 \leqslant j \leqslant n, \tag{2.1}$$

where $V(\zeta) \equiv (V_1(\zeta), \dots, V_n(\zeta))$. The action of

$$\Delta_{V,Y} : D_{V,Y} \subset L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$$

is then given by

$$\Delta_{V,Y} \phi := \Delta \phi_{\mathrm{reg}}.$$

The set Y is the singular set of the point interaction. It is the set where the elements of the domain of $\Delta_{V,Y}$ do not belong to $\bar{H}^2(\mathbb{R}^3)$, or better, since $\bar{H}^2(\mathbb{R}^3) \subset C_b(\mathbb{R}^3)$, where they are unbounded.

Let us define, for any $z \in \mathbb{C} \backslash (-\infty, 0]$,

$$G_i^z(x) := \frac{\exp(-\sqrt{z}|x - y_i|)}{4\pi |x - y_i|}, \qquad \mathrm{Re}\sqrt{z} > 0,$$

and

$$(M_Y(z))_{ij} := (1 - \delta_{ij}) G_i^z(y_j), \qquad \left\langle G_Y^z, \phi \right\rangle_i := \langle G_i^z, \phi \rangle.$$

Then one has the following.

Lemma 2.2. *For any $z \in \mathbb{C} \backslash (-\infty, 0]$ such that the function*

$$\Gamma_{V,Y}(z) : A_V \to \mathbb{C}^n, \qquad \Gamma_{V,Y}(z) := V + \frac{\sqrt{z}}{4\pi} - M_Y(z)$$

has an inverse, the nonlinear resolvent of $\Delta_{V,Y}$ is given by

$$(-\Delta_{V,Y} + z)^{-1} \phi = (-\Delta + z)^{-1} \phi + \sum_{1 \leqslant i \leqslant n} \left(\Gamma_{V,Y}(z)^{-1} \left\langle G_Y^{\bar{z}}, \phi \right\rangle \right)_i G_i^z.$$

Proof. We need to solve the equation $(-\Delta_{V,Y} + z)\psi = \phi$. By the definition of $\Delta_{V,Y}$ one has

$$\psi_{\mathrm{reg}} = (-\Delta + z)^{-1} \phi - z \sum_{1 \leqslant i \leqslant n} \zeta_i^\psi (-\Delta + z)^{-1} G_i$$

and

$$\psi_{\mathrm{reg}}(y_j) = ((V - M_Y(0))(\zeta^\psi))_j = \left\langle G_j^{\bar{z}}, \phi \right\rangle - z \sum_{1 \leqslant i \leqslant n} \zeta_i^\psi \left\langle G_j^{\bar{z}}, G_i \right\rangle.$$

Since $z\left\langle G_j^{\bar{z}}, G_i \right\rangle = (M_Y(0) - M_Y(z))_{ij}$ and $z\left\langle G_i^{\bar{z}}, G_i \right\rangle = 1/4\pi\sqrt{z}$, one obtains

$$\zeta^\phi = \Gamma_{V,Y}(z)^{-1} \left\langle G_Y^z, \phi \right\rangle$$

so that

$$\psi = (-\Delta + z)^{-1}\phi + \sum_{1\leqslant i\leqslant n} \left(\Gamma_{V,Y}(z)^{-1}\langle G_Y^{\bar z}, \phi\rangle\right)_i (G_i - z(-\Delta + z)^{-1}G_i)$$
$$= (-\Delta + z)^{-1}\phi + \sum_{1\leqslant i\leqslant n} \left(\Gamma_{V,Y}(z)^{-1}\langle G_Y^{\bar z}, \phi\rangle\right)_i G_i^z. \qquad \square$$

Remark 2.3. The nonlinear resolvent $R_{V,Y}(z) := (-\Delta_{V,Y} + z)^{-1}$ satisfies the nonlinear resolvent identity

$$R_{V,Y}(z) = R_{V,Y}(w)(1 - (z - w)R_{V,Y}(w)).$$

Thus $\Delta_{V,Y}$ can be alternatively defined as

$$\Delta_{V,Y}\phi := (-R_{V,Y}(z)^{-1} + z)\phi = \Delta\phi_z + z \sum_{1\leqslant j\leqslant n} \zeta_j^\phi G_j^z$$

on

$$D_{V,Y} := \mathrm{Range}(R_{V,Y}(z)) = \Bigg\{\phi \in L^2(\mathbb{R}^3) : \phi = \phi_z + \sum_{1\leqslant j\leqslant n} \zeta_j^\phi G_j^z, \phi_z \in H^2(\mathbb{R}^3),$$
$$\Gamma_{V,Y}(z)\zeta^\phi = (\phi_z(y_1), \ldots, \phi_z(y_N))\Bigg\},$$

the definition being z-independent. In the above definition $H^2(\mathbb{R}^3) := \bar H^2(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ denotes the usual Sobolev space of index 2. For future convenience, we also introduce the Sobolev spaces of index 1,

$$\bar H^1(\mathbb{R}^3) := \left\{f \in L^2_{\mathrm{loc}}(\mathbb{R}^3) : \nabla f \in L^2(\mathbb{R}^3)\right\},$$

and $H^1(\mathbb{R}^3) := \bar H^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$.

Lemma 2.2 also shows that when V is linear and represented by a Hermitian matrix Θ, the linear operator $\Delta_{\Theta,Y}$ coincides with the self-adjoint operator giving the usual Laplacian with n point interactions placed at Y (see [3, 4]).

The form domain of the operator $\Delta_{\Theta,Y}$, which we denote by $\dot D_Y$, is the set of $\phi \in L^2(\mathbb{R}^3)$ for which there exists an n-tuple of complex numbers $\zeta^\phi = \left(\zeta_1^\phi, \ldots, \zeta_n^\phi\right)$ such that $\phi_{\mathrm{reg}} \in \bar H^1(\mathbb{R}^3)$, where $\phi_{\mathrm{reg}} \in L^2_{\mathrm{loc}}(\mathbb{R}^3)$ is defined as before. Note that here no restriction at all is imposed on the vector ζ^ϕ so that $D_{V,Y} \subset \dot D_Y$. The quadratic form corresponding to the linear operator $-\Delta_{\Theta,Y}$ is then given by

$$\mathcal{F}_{\Theta,Y}(\phi) = \|\nabla\phi_{\mathrm{reg}}\|^2_{L^2} - (M_Y\zeta^\phi, \zeta^\phi) + (\Theta\zeta^\phi, \zeta^\phi)$$

(see [18]), where $(\cdot, \cdot)$ denotes the usual Hermitian scalar product on $\mathbb{C}^n$ and M_Y is the symmetric matrix $M_Y := M_Y(0)$.

3. Existence and uniqueness

Theorem 3.1. *Let $V : A_V \subseteq \mathbb{C}^n \to \mathbb{C}^n$ be Lipschitz, let $\phi_0 \in D_{V,Y}$ and $\dot\phi_0 \in \dot D_Y$. Let $\zeta(t)$, $t \in (-T, T)$, be the unique maximal solution of the Cauchy problem with delay*

$$\frac{\mathrm{sgn}(t)}{4\pi}\dot\zeta_j(t) + V_j(\zeta(t)) = \sum_{i\neq j} \frac{\theta(|t| - |y_i - y_j|)}{4\pi|y_i - y_j|}\zeta_i(t - \mathrm{sgn}(t)|y_i - y_j|) + \phi_f(t, y_j),$$
$$\zeta(0) = \zeta^{\phi_0}, \quad 1 \leqslant j \leqslant n, \tag{3.1}$$

where θ denotes the Heaviside function and ϕ_f is the unique solution of the Cauchy problem,

$$\ddot{\phi}_f(t) = \Delta\phi_f(t), \qquad \phi_f(0) = \phi_0, \qquad \dot{\phi}_f(0) = \dot{\phi}_0. \tag{3.2}$$

Defining, given $s \in \mathbb{R}$,

$$\phi(t,x) := \phi_f(t-s,x) + \sum_{1\leqslant j\leqslant n} \frac{\theta(|t-s| - |x-y_j|)}{4\pi|x-y_j|}\zeta_j((t-s) - \mathrm{sgn}(t-s)|x-y_j|),$$

one has

$$\phi(t) \in D_{V,Y}, \qquad \dot{\phi}(t) \in \dot{D}_Y, \qquad \zeta^{\phi(t)} = \zeta(t-s), \qquad \zeta^{\dot{\phi}(t)} = \dot{\zeta}(t-s)$$

for all $t \in (-T+s, T+s)$ and ϕ is the unique strong solution of the Cauchy problem,

$$\ddot{\phi}(t) = \Delta_{V,Y}\phi(t), \qquad \phi(s) = \phi_0, \qquad \dot{\phi}(s) = \dot{\phi}_0. \tag{3.3}$$

Moreover, defining the nonlinear map

$$U_{V,Y}(t) : D_{V,Y} \times \dot{D}_Y \to D_{V,Y} \times \dot{D}_Y, \qquad t \in (-T,T),$$
$$U_{V,Y}(t)(\phi_0, \dot{\phi}_0) := (\phi(t+s), \dot{\phi}(t+s)),$$

one has, for any $t_1, t_2 \in (-T,T)$ with $t_1 + t_2 \in (-T,T)$, the group property

$$U_{V,Y}(t_1)U_{V,Y}(t_2) = U_{V,Y}(t_1+t_2). \tag{3.4}$$

We preface to the proof some preparatory lemmata:

Lemma 3.2. *Let $\xi : (a,b) \to \mathbb{C}$. Then*

$$\xi \in L^2_{\mathrm{loc}}(a,b) \iff \forall t \in (a,b), \quad \psi(t) \in H^2(\mathbb{R}^3),$$
$$\xi \in L^2_{\mathrm{loc}}(a,b) \iff \forall t \in (a,b), \quad \dot{\psi}(t) \in H^1(\mathbb{R}^3),$$

where ψ is the unique solution of $\ddot{\psi}(t) = \Delta\psi(t) + \xi(t)G_i^1$ with zero initial data.

Proof. Since the unique solution of $\ddot{\varphi}(t) = \Delta\varphi(t) + \xi(t)\delta_{y_i}$ with zero intitial data (at time $t = 0$) is given by

$$\varphi(t,x) = \frac{\theta(|t| - |x-y_i|)}{4\pi|x-y_i|}\xi(t - \mathrm{sgn}(t)|x-y_i|)$$

and $G_i^1 = (-\Delta+1)^{-1}\delta_{y_i}$, we have $(-\Delta+1)\psi = \varphi$. Thus

$$\|(-\Delta+1)\psi(t)\|^2_{L^2} = \frac{1}{4\pi}\int_0^{|t|} \mathrm{d}r|\xi(t - \mathrm{sgn}(t)r)|^2$$
$$= \frac{1}{4\pi}\begin{cases} \int_0^t \mathrm{d}s|\xi(s)|^2, & t > 0 \\ \int_t^0 \mathrm{d}s|\xi(s)|^2, & t < 0. \end{cases}$$

Since, by Fourier transform (we suppose $t > 0$, the case $t < 0$ is analogous),

$$\sqrt{|k|^2+1}\,\hat{\dot{\psi}}(t) = \frac{1}{(2\pi)^{3/2}}\frac{1}{\sqrt{|k|^2+1}}\int_0^t \mathrm{d}s\, \xi(s)\cos(t-s)|k|,$$

one has

$$\|\sqrt{-\Delta+1}\dot{\psi}(t)\|^2_{L^2} = \frac{1}{2\pi^2}\lim_{R\uparrow\infty}\int_0^t\int_0^t \mathrm{d}s\,\mathrm{d}s'\,\bar{\xi}(s)\xi(s')\int_0^R \mathrm{d}r\frac{r^2\cos(t-s)r\cos(t-s')r}{r^2+1}$$
$$= \frac{1}{4\pi}\int_0^t \mathrm{d}s|\xi(s)|^2 + \frac{1}{8\pi}\int_0^t\int_0^t \mathrm{d}s\,\mathrm{d}s'\,\bar{\xi}(s)\xi(s')(\mathrm{e}^{-|s-s'|} + \mathrm{e}^{-2t}\,\mathrm{e}^{-(s+s')})$$
$$\leqslant \left(\frac{1}{4\pi} + t^2\frac{1+\mathrm{e}^{-2t}}{8\pi}\right)\int_0^t \mathrm{d}s|\xi(s)|^2.$$

Conversely, $\|\sqrt{-\Delta+1}\dot{\psi}(t)\|_{L^2}^2$ for all $t \in (a,b)$ implies $\xi \in L^2_{\text{loc}}(a,b)$ since the second term in the last equality above is positive. □

Lemma 3.3. *Let φ_i be the solutions of the free wave equation with initial data $\varphi_i(0) = \zeta_i G_i$, $\dot{\varphi}_i(0) = \dot{\zeta}_i G_i$. Then*

$$\varphi_i(t, y_i) = \frac{\text{sgn}(t)}{4\pi}\dot{\zeta}_i.$$

Proof. Since $\psi(t) := \varphi_i(t) - (\zeta_i + t\dot{\zeta}_i)G_i$ satisfies

$$\ddot{\psi}(t) = \Delta\psi(t) - (\zeta_i + t\dot{\zeta}_i)\delta_{y_i}$$

with zero initial data, one obtains

$$\varphi_i(t,x) = -\frac{\theta(|t|-|x-y_i|)(\zeta_i + (t-\text{sgn}(t)|x-y_i|)\dot{\zeta}_i)}{4\pi|x-y_i|} + \frac{\zeta_i + t\dot{\zeta}_i}{4\pi|x-y_i|},$$

and the proof is completed by taking the limit $x \to y_i$. □

Lemma 3.4. *Let φ be the solution of the free wave equation with regular initial data $\varphi(0) \in \bar{H}^2(\mathbb{R}^3)$ and $\dot{\varphi}(0) \in \bar{H}^1(\mathbb{R}^3)$. Then for all $y \in \mathbb{R}^3$ there exists $\zeta_y \in C^1(\mathbb{R})$ with $\ddot{\zeta}_y \in L^2_{\text{loc}}(\mathbb{R})$ such that*

$$\varphi(t,y) = \frac{\text{sgn}(t)}{4\pi}(\dot{\zeta}_y(t) - \dot{\zeta}_y(0)) + \zeta_y(t).$$

Moreover,

$$\lim_{|t|\uparrow\infty}\varphi(t,y) = 0.$$

Proof. Let us consider the linear operator $\Delta_{1,y}$ corresponding to $Y = \{y\}$ and $V = 1$. Then, by the results in [14, section 3] (also see [12], theorem 3), theorem 3.1 holds true for $\Delta_{1,y}$, with $\phi \in C^0(\mathbb{R}, D_{1,y}) \cap C^1(\mathbb{R}, \dot{D}_y) \cap C^2(\mathbb{R}, L^2(\mathbb{R}))$. This implies that, by lemma 2.3, $\psi(t) := \phi(t) - \zeta^\phi(t)G^1_y$ belongs to $H^2(\mathbb{R}^3)$ for all t and that $\zeta_y \equiv \zeta^\phi \in C^1(\mathbb{R})$ since $\dot{D}_y$ is normed by $\|\phi\|^2_{\dot{D}_y} := \|\nabla\phi_{\text{reg}}\|^2_{L^2} + |\zeta^\phi|^2$. Since $\psi(t)$ solves the equation $\ddot{\psi}(t) = \Delta\psi(t) - \ddot{\zeta}_y(t)G^1_y$ with initial data $\psi(0) \in H^2(\mathbb{R}^3)$ and $\dot{\psi}(0) \in H^1(\mathbb{R}^3)$, $\ddot{\zeta}_y \in L^2_{\text{loc}}(\mathbb{R})$ by lemma 3.2. Moreover, ζ_y solves the differential equation

$$\frac{\text{sgn}(t)}{4\pi}\dot{\zeta}_y(t) + \zeta_y(t) = \phi_f(t,y)$$

so that, by lemma 3.3,

$$\varphi(t,y) = \frac{\text{sgn}(t)}{4\pi}(\dot{\zeta}_y(t) - \dot{\zeta}_y(0)) + \zeta_y(t).$$

The fact that $\varphi(t,y) \to 0$ as $|t| \uparrow \infty$ follows from the well-known decay properties of the solution of the free wave equation with regular initial data. □

Remark 3.5. The two previous lemmata show that $\phi_f(t, y_j)$ in (2.1) is made of two pieces: a continuous and bounded one and another which has a jump of size $\frac{\zeta^{\dot{\phi}_0}}{2\pi}$ at the origin. Thus, by taking the limit $t \to \pm 0$ in (3.1), $\dot{\zeta}(0_-) = \dot{\zeta}(0_+) = \zeta^{\dot{\phi}_0}$ and the forward and backward solutions match together at the initial time.

Proof of theorem 3.1. Let

$$\psi(t) := \phi(t) - \sum_{1\leqslant i\leqslant n}\zeta_i(t-s)G^1_i.$$

Since ζ solves (3.1) and $\phi_f(\cdot, y_i)$ is almost everywhere derivable with a derivative in $L^2_{\text{loc}}(\mathbb{R})$ by lemmas 3.3 and 3.4, one has that ζ is piecewise C^1 with $\dot\zeta \in L^\infty(-T, T)$ and $\ddot\zeta \in L^2_{\text{loc}}((-T, T))$. Thus $\psi(t) \in H^2(\mathbb{R}^3)$ and $\dot\psi(t) \in H^1(\mathbb{R}^3)$ for all $t \in (-T+s, T+s)$ by lemma 3.2 since

$$\ddot\psi(t) = \Delta\psi(t) - \sum_{1\leqslant i\leqslant n} \ddot\zeta_i(t-s)G_i^1.$$

Since $G_i^1 - G_i \in \bar H^2(\mathbb{R}^3)$, this implies $\phi_{\text{reg}}(t) \in \bar H^2(\mathbb{R}^3)$ and $\dot\phi_{\text{reg}} \in \bar H^1(\mathbb{R}^3)$, where

$$\phi_{\text{reg}}(t) := \phi(t) - \sum_{1\leqslant i\leqslant n} \zeta_i(t-s)G_i$$

and

$$\dot\phi_{\text{reg}}(t) := \dot\phi(t) - \sum_{1\leqslant i\leqslant n} \dot\zeta_i(t-s)G_i.$$

Thus $\dot\phi(t) \in \dot D_Y$ and $\zeta^{\dot\phi(t)} = \dot\zeta(t-s)$. Moreover $\phi(t) \in D_{V,Y}$, with $\zeta^\phi(t) \equiv \zeta(t-s)$ if the boundary conditions (2.1) hold true for all $t \in (-T+s, T+s)$. Since ζ solves (3.1) one has

$$\begin{aligned}\lim_{x\to y_j}(\phi(t,x) - \zeta_j(t-s)G_j) &= \phi_f(t-s, y_j) + \sum_{i\neq j}\frac{\theta(|t-s| - |y_i - y_j|)}{4\pi|y_j - y_i|}\zeta_i((t-s) \\ &\quad - \operatorname{sgn}(t-s)|y_i - y_j|) + \frac{\zeta_j((t-s) - \operatorname{sgn}(t-s)|x-y_j|) - \zeta_j(t-s)}{4\pi|x-y_j|} \\ &= \phi_f(t-s, y_j) + \sum_{i\neq j}\frac{\theta(|t-s| - |y_i - y_j|)}{4\pi|y_j - y_i|}\zeta_i((t-s) \\ &\quad - \operatorname{sgn}(t-s)|y_i - y_j|) - \frac{\operatorname{sgn}(t-s)}{4\pi}\dot\zeta_j(t-s) = V_j(\zeta(t-s))\end{aligned}$$

and (2.1) are satisfied. Once we know that $\phi(t) \in D_{V,Y}$, one has

$$\ddot\phi = \Delta\phi + \sum_{1\leqslant j\leqslant n}\zeta_j\delta_{y_j} = \Delta\left(\phi - \sum_{1\leqslant j\leqslant n}\zeta_j G_j\right) \equiv \Delta_{V,Y}\phi_{\text{reg}}$$

and so ϕ solves (3.3).

Suppose now that φ is another strong solution of (3.3). Then, by reversing the above argument, the boundary conditions (2.1) imply that ζ^ϕ solves the Cauchy problem (3.1). By unicity of the solution of (3.1) one obtains $\zeta^\phi(t) = \zeta(t-s)$. Then, defining

$$\varphi_f(t) := \varphi(t) - \sum_{1\leqslant j\leqslant n}\phi_j(t-s),$$

where

$$\phi_j(t,x) := \frac{\theta(|t| - |x-y_j|)}{4\pi|x-y_j|}\zeta_j(t - \operatorname{sgn}(t)|x-y_j|),$$

one obtains

$$\ddot\varphi_f = \Delta\varphi_{\text{reg}} - \sum_{1\leqslant j\leqslant n}\left(\Delta\phi_j + \zeta_j\delta_{y_j}\right) = \Delta\left(\varphi_{\text{reg}} - \sum_{1\leqslant j\leqslant n}(\phi_j - \zeta_j G_j)\right) = \Delta\varphi_f,$$

i.e., φ_f solves the Cauchy problem (3.2). Thus, by unicity of the solution of (3.2), $\varphi = \phi$.

The proof of (3.4) is standard: by considering the first components of $U_{V,Y}(t)U_{V,Y}(t_1)(\phi_0, \dot\phi_0)$ and $U_{V,Y}(t+t_1)(\phi_0, \dot\phi_0)$ (with $t \in [0, t_2]$) one obtains two strong solutions of (3.3). They coincide by unicity and so (3.4) holds true. $\square$

Remark 3.6. By proceeding as in the linear case (see [12]) one can show that the wave equation $\ddot{\phi} = \Delta_{V,Y}\phi$ has finite speed of propagation if and only if $V_j(\zeta) = V_j(\zeta_j)$ for all j.

When the vector field V is of gradient type, the flow $U_{V,Y}(t)$ preserves an energy-like quantity:

Lemma 3.7. *If* $V = \nabla h$ *then*

$$\forall t \in (-T, T), \qquad \mathcal{E}_{V,Y} U_{V,Y}(t) = \mathcal{E}_{V,Y},$$

where the energy $\mathcal{E}_{V,Y}$ *is defined by*

$$\mathcal{E}_{V,Y}(\phi, \dot{\phi}) := \tfrac{1}{2}\big(\|\dot{\phi}\|_{L^2}^2 + \|\nabla\phi_{\mathrm{reg}}\|_{L^2}^2 - (M_Y\zeta^{\phi}, \zeta^{\phi})\big) + \mathrm{Re}(h(\zeta^{\phi})).$$

Proof.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|\dot{\phi}\|_2^2 &= \langle \Delta_{Y,V}\phi, \dot{\phi}\rangle + \langle \dot{\phi}, \Delta_{Y,V}\phi\rangle \\
&= \left\langle \Delta\phi_{\mathrm{reg}}, \dot{\phi}_{\mathrm{reg}} + \sum_{1\leqslant i\leqslant n} \dot{\zeta}_i^{\phi} G_i \right\rangle + \left\langle \dot{\phi}_{\mathrm{reg}} + \sum_{1\leqslant i\leqslant n} \dot{\zeta}_i^{\phi} G_i, \Delta\phi_{\mathrm{reg}} \right\rangle \\
&= \langle \Delta\phi_{\mathrm{reg}}, \dot{\phi}_{\mathrm{reg}}\rangle - \sum_{1\leqslant i\leqslant n} \dot{\zeta}_i^{\phi} \bar{\phi}_{\mathrm{reg}}(y_i) + \left\langle \dot{\phi}_{\mathrm{reg}}, \Delta\phi_{\mathrm{reg}} - \sum_{1\leqslant i\leqslant n} \dot{\bar{\zeta}}_i^{\phi} \phi_{\mathrm{reg}}(y_i) \right\rangle \\
&= \langle \Delta\phi_{\mathrm{reg}}, \dot{\phi}_{\mathrm{reg}}\rangle + (M_Y\zeta^{\phi}, \dot{\zeta}^{\phi}) - \sum_{1\leqslant i\leqslant n} \dot{\zeta}_i^{\phi} \bar{V}_i(\zeta^{\phi}) \\
&\quad + \langle \dot{\phi}_{\mathrm{reg}}, \Delta\phi_{\mathrm{reg}}\rangle + (\dot{\zeta}^{\phi}, M_Y\zeta^{\phi}) - \sum_{1\leqslant i\leqslant n} \dot{\bar{\zeta}}_i^{\phi} V_i(\zeta^{\phi}) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\big(-\|\nabla\phi_{\mathrm{reg}}\|_2^2 + (M_Y\zeta^{\phi}, \zeta^{\phi}) - 2\,\mathrm{Re}(h(\zeta^{\phi}))\big).
\end{aligned}$$

□

The above conservation result can be used to obtain a global existence result by standard arguments:

Theorem 3.8. *Let* $V = \Theta + \nabla h$ *with* Θ *a Hermitian matrix and* h *such that*

$$\mathrm{Re}(h(\zeta)) \geqslant c_1|\zeta|^2 - c_2, \qquad c_1 \geqslant 0, \quad c_2 \geqslant 0.$$

Then the flow $U_{V,Y}(t)$ *is global.*

Proof. In this case $\mathcal{E}_{V,Y}(\phi, \dot{\phi}) = \frac{1}{2}\big(\|\dot{\phi}\|_2^2 + \mathcal{F}_{\Theta,Y}(\phi)\big) + \mathrm{Re}(h(\zeta))$. Since V is of gradient type and $\mathcal{F}_{\Theta,Y}$ is bounded from below (see [3]),

$$|\zeta^{\phi}(t)|^2 \leqslant k\mathcal{E}_{V,Y}(\phi(t), \dot{\phi}(t)) = k\mathcal{E}_{V,Y}(\phi(0), \dot{\phi}(0))$$

for some positive constant k. □

4. Blowing-up solutions and their lifespan

The solution given in theorem 3.1 can be obviously extended to the spacetime domain

$$E = \bigcap_{y\in Y}\{(t, x) \in \mathbb{R}^4 : -T + s - |x - y| < t < T + s + |x - y|\}.$$

Such a function is a local solution on E in the sense that

$$\partial^2_{tt}\phi(t,x) = \Delta\left(\phi - \sum_{1\leqslant j\leqslant n}\zeta^{\phi}_j G_j\right)(t,x) \qquad (t,x)\in E \qquad \phi(s)=\phi_0 \qquad \dot{\phi}(s)=\dot{\phi}_0.$$

Note, however, that no boundary conditions can be imposed on ϕ at times $t\notin(-T+s, T+s)$, since $\{t\}\times Y$ is not included in E when $t\notin(-T+s, T+s)$.

If $\zeta(t)$ blows up at times $\pm T$ then the above local solution has a blow-up boundary given by ∂E.

Now we turn to the detailed analysis of the case in which $Y=\{y\}$, so that there is no delay in (3.1). Since (by direct verification) the backward solution ζ_- of (3.1) is related to a forward solution by $\zeta_-(t)=\zeta_+^-(-t)$, where ζ_+^- is the forward solution of (3.1) with inhomogeneous term $\phi_f(-t,y)$, we will concentrate the analysis on the solutions of the system

$$\dot{\zeta}(t)+V(\zeta(t))=g(t) \qquad \zeta(0)=\zeta_0$$

where $g(t)$ is continuous and $g(t)=g_0+g_1(t)$ with $g_1(t)\to 0$ as $|t|\uparrow\infty$ (see remark 2.5). We will study the equation in a real framework, i.e. we suppose that the fields ϕ and V are real-valued. Hence $\zeta(t)\in\mathbb{R}$. Moreover, to fix ideas, let us consider a continuous function $V:\mathbb{R}\to\mathbb{R}$, and regular enough to ensure local existence and unicity of the solution of the differential equation.

Let us begin by considering preliminarily the case in which $g(t)=g_0$ is constant. This gives the autonomous differential equation $\dot{z}+V(z)=g_0$ with equilibrium (constant) solutions given by the z which satisfy the equation $V(z)=g_0$. Let us fix an initial datum $z_0=z(0)$ not belonging to such a set. Correspondingly, in the interval of existence of the solution the term $g_0-V(z(t))$ has constant sign by continuity, and the solution $z(t)$ is implicitly given by the relation

$$\int_{z_0}^{z(t)}\frac{\mathrm{d}s}{g_0-V(s)}=t.$$

This implies an elementary but fundamental remark. The solution of the auxiliary equation is global if and only if both the improper integrals

$$\int_{z_0}^{\pm\infty}\frac{\mathrm{d}s}{g_0-V(s)}$$

diverge. If, in contrast, at least one of them converges, the solution blows up in the past or in the future and the backward or forward lifespan $T_\pm$ of the solution is given just by the value of one of such integrals.

Now the main point is to include in the analysis the time-dependent term $g_1(t)$. The presence of this term is an essential preclusion to the possibility of writing down a closed formula for the solution of our differential equation, and one has to resort to other methods. A first remark is that the time-dependent term $g_1(t)$ is bounded. This suggests that the behaviour of the solution of the inhomogeneous equation could not be affected so much by this term. The idea is to use differential inequalities to confront the size of solutions of the two equations. Roughly speaking, a solution greater or lower than a function blowing up to $+\infty$ or $-\infty$, respectively, is blowing up; and a solution which is bounded between two functions finite at every finite time is global in time. Both situations occur, and both can occur for the same coupling depending on the initial data. The analysis is based on differential inequalities which relate the solution of a comparison auxiliary equation with the solution of the given equation. We recall briefly that the defect operator P associated with the differential equation $\dot{z}(t)=F(t,z(t))$ is given by

$$P(t,z):=\dot{z}-F(t,z(t)).$$

Comparison of defect operator leads to important and classical differential inequalities (see, e.g., [19]):

Theorem 4.1. *Let* $z_-(0) \leqslant z(0) \leqslant z_+(0)$ *and*

$$P(t, z_-) \leqslant 0 = P(t, z) \leqslant P(t, z_+), \qquad t \in [a, b].$$

Then one has $z_-(t) \leqslant z(t) \leqslant z_+(t)$ *in* $[a, b]$.

Correspondingly, with a terminology introduced by Perron, z_- is called a subsolution and z_+ is called a supersolution.

Now, let us define $K := \sup_{t\in\mathbb{R}}|g(t)|$ and consider the couple of differential equations

$$\dot{z}_\pm(t) + V(z_\pm(t)) = \pm K$$

with initial conditions $z_-(0) \leqslant \zeta_0$ and $z_+(0) \geqslant \zeta_0$. It is immediate to see that one has the following inequalities between defect operator:

$$P(z_-, t) = g(t) - K \leqslant 0 = P(t, \zeta) \leqslant g(t) + K = P(t, z_+).$$

So z_- is a subsolution and z_+ is a supersolution of ζ. Of course to a subsolution z_- positively blowing up in the future with a lifespan T_+ corresponds to a solution ζ positively blowing up with a lifespan $T_* < T_+$. A similar reasoning applies to negatively blowing up supersolutions. Since by theorem 3.1 $U_{V,Y}(-t) = U_{V,Y}(t)^{-1}$, we do not take account of solutions blowing up in the past. Summarizing, we obtain the following criterion for global existence or blow-up:

Theorem 4.2. *Let* $\phi(t)$ *be the solution of the Cauchy problem (3.3) with* $\zeta_0 = \zeta^{\phi(0)}$ *and put* $K := \sup_{t\in\mathbb{R}}|\phi_f(t, y)|$*:*

(1) ϕ *is a global solution if the integrals*

$$\int_{\zeta_0}^{\pm\infty} \frac{\mathrm{d}s}{K + V(s)}, \qquad \int_{\zeta_0}^{\pm\infty} \frac{\mathrm{d}s}{K - V(s)}$$

diverge;

(2) ϕ *is positively blowing up in the future if*

$$-\int_{\zeta_0}^{+\infty} \frac{\mathrm{d}s}{K + V(s)}$$

converges to a positive value. The value of such an integral gives an upper bound of the lifespan T_**. An analogous statement holds true for solutions negatively blowing up in the future.*

Remark 4.3. Note that the constant K depends on both $\phi(0)$ and $\dot{\phi}(0)$. So as it should be expected, the complete set of initial data determines the global existence or blow-up of the solution.

Remark 4.4. Another very simple criterion of global existence is the following: suppose that both the sets $S_\pm = \{s_\pm : V(s) = \pm K\}$ are not void. Then any $s_\pm \in S_\pm$ provide global (stationary) super- and subsolutions. This gives global existence for solutions with $s_- \leqslant \zeta_0 \leqslant s_+$.

4.1. Examples

Typical nonlinearities in model equations are given by a power law, or more generally polynomial couplings. They are essentially to be considered as phenomenological choices, typically originated by some *ad hoc* truncation of a Taylor approximation of more general couplings.

Let us consider the function

$$V(\zeta) = \gamma|\zeta|^\sigma\zeta, \qquad \gamma > 0, \quad \sigma \in \mathbb{R}.$$

The auxiliary equations

$$\dot{z}_\pm = -\gamma|z_\pm|^\sigma z_\pm \pm K$$

have the equilibrium solutions $s_\pm = \pm\left(\frac{K}{\gamma}\right)^{\frac{1}{\sigma+1}}$. Thus, by remark 4.4, one has a global solution for any initial data with $|\zeta_0| \leqslant \left(\frac{K}{\gamma}\right)^{\frac{1}{\sigma+1}}$. For data with $|\zeta_0| > \left(\frac{K}{\gamma}\right)^{\frac{1}{\sigma+1}}$ and for $\sigma > 0$, the integral

$$\int_{\zeta_0}^{\infty} \frac{\mathrm{d}s}{\gamma|s|^\sigma s + K}$$

converges and so in this case we have blow-up with a lifespan

$$|T_*| < \int_{\zeta_0}^{\infty} \frac{\mathrm{d}s}{\gamma|s|^\sigma s + K}.$$

A nonlinearity which deserves attention is given by $V(\zeta) = \alpha\zeta^2$. This admits an analysis analogous to the one just devised and corresponds to a quadratic nonlinearity in the abstract wave equation. The peculiarity is that in this case (3.1) is a Riccati equation, one of the better known nonlinear differential equations of the first order and one with wide applications in mathematics and sciences. One of the more striking properties of the Riccati equation is that by a nonlinear transformation of the unknown function, it can be reduced to a second-order linear differential equation, and this fact appears as particularly noteworthy in our context, where the original problem is a wave equation with a quadratic (concentrated) nonlinearity. We were not able, till now, to judge about the relevance of this fact, which seems to deserve further investigation. Another well-known property of Riccati equation is the fact that it admits always at least one nonglobal solution (see, e.g., [10]) when the time-dependent term $g(t)$ is an algebraic function. Of course, there is no hope that the evaluation at y of a solution of a free wave equation be an algebraic equation, at least for generic data, but in our case, thanks to the properties of $\phi_f(t, y)$, the simple majorizations above allow us to obtain blowing-up solution also in the case of nonalgebraic inhomogeneous terms. Moreover, another important fact about blowing-up solutions of the Riccati equation is the typical behaviour of the solution in the proximity of the blow-up time T_* lifespan, which is of the type

$$\zeta(t) \sim \frac{1}{t - T_*}.$$

This gives, in view of the relation between the time behaviour of the $\zeta(t)$ and the behaviour of the solution, the qualitative asymptotic spacetime behaviour of the solution of the wave equation with quadratic concentrated nonlinearity near the blow-up time, which is of the type

$$\phi(t, x) \sim \frac{1}{t - T_* - |x - y|}\frac{1}{|x - y|}.$$

Similar consideration holds for the power nonlinearities analysed above, or other non-polynomial couplings for which a precise analysis of the $\zeta(t)$ equation is feasible.

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